Landau damping in a plasma.

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September 24, 1969

Abstract

This article gives a short review of the phenomenon Landau damping.[1] The theoretical foundation is shortly discussed. Nonlinear Landau damping is discussed from a phenomenological point of view. Reversibility and collision free damping is discussed in connection to the echo phenomenon.

1 Introduction

The first studies on plasma oscillations were done by Tonks and Langmuir 1925 [2]. Thy used modified MHD-equations together with Poisson's equation, in order to describe these plasma oscillations, also called Langmuir oscillations, especially in Russian literature. Vlasov in 1938 studied the same phenomenon using kinetic equations and a self consistent electric field. Also calle Vlasov's equation.

2 Vlasov theory

Vlasov equation reads

$$\frac{\partial F}{\partial t} + \mathbf{v} \cdot \nabla F - \frac{q\mathbf{E}}{m} \cdot \nabla_{\mathbf{v}} F = 0 \ . \tag{1}$$

Here $F = F(\mathbf{r}, \mathbf{v}, t)$ is the one particle distribution function, where $\mathbf{r}$ is the position vector, $\mathbf{v}$ is the velocity vector and t the time. $\mathbf{E}$ is the self consistent electric field determined by the Poisson's equation

$$\nabla \cdot \mathbf{E} = 4\pi q \cdot [n - \int d^3\mathbf{v} F(\mathbf{r}, \mathbf{v}, t)] \ . \tag{2}$$

Furthermore m is the electron mass, q the electron charge and n the ion density, which is regarded being constant for the high frequency plasma oscillations.

[1]This work refers to a lecture given for the Dr. philos degree (PHD) at The University of Bergen October 24, 1969. On this occasion two lectures were required, one with a topic you chose yourself, and another with a topic given to you by the committee that were appointed for judging the work, a topic for this lecture should be complimentary to the work familiar to the candidate. The given topic in my case was: "Landau damping in a plasma". The following account is a retyping of the note prepared for this lesson, which was written in the Norwegian language at the time. It is here translated into the English language.

As usual we introduce the electric potential by

$$\mathbf{E} = -\nabla\phi \; . \tag{3}$$

We shall assume an equilibrium solution which is uniformly valid with $F = f_0'$ and $\phi = \phi_0 = 0$. We assume small oscillations around this equilibrium and describe them with $F = f_0' + f'$ and ϕ. We shall omit the nonlinear term

$$\frac{q\nabla\phi}{m} \cdot \nabla_v f' \tag{4}$$

in Eq(1) and obtain

$$\frac{\partial f'}{\partial t} + \mathbf{v} \cdot \nabla f' + \frac{q}{m} \nabla\phi \cdot \nabla_v f_0' = 0 \; , \tag{5}$$

$$\nabla^2 \phi = 4\pi q \int d^3 f'(\mathbf{v}, \mathbf{r}, t) \; . \tag{6}$$

We shall consider plane waves as $\exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$. Then from Eq. (5) and (6) we find

$$i\omega f' + i\mathbf{v} \cdot \mathbf{k} f' + \frac{q}{m} i\phi \mathbf{k} \cdot \nabla_v f_0' = 0 \; , \tag{7}$$

$$-k^2 \phi = 4\pi q \int d^3 f'(\mathbf{v}, \mathbf{r}, t) \; . \tag{8}$$

We now decompose $\mathbf{v}$ into parallel and perpendicular components $\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp$ where $\mathbf{v}_\parallel = \frac{\mathbf{k}}{k} \cdot \mathbf{v} \frac{\mathbf{k}}{k}$. By integrating over $\mathbf{v}_\perp$ in velocity space and introducing

$$\int_{-\infty}^{+\infty} d^2\mathbf{v}_\perp f'(\mathbf{v}, \mathbf{r}, t) = f(v_\parallel, \mathbf{r}, t) \quad \text{and} \quad \int_{-\infty}^{+\infty} d^2\mathbf{v}_\perp f_0' = f_0 \; . \tag{9}$$

we obtain from Eqs.(8) and (9)

$$-(\omega - kv) f = -\frac{qk}{m} \phi \frac{\partial f_0}{\partial v} \; . \tag{10}$$

Where we have now put $v_\parallel = v$ since this now can not be misinterpreted and $f' = f$. Furthermore

$$-k^2 \phi = 4\pi q \int_{-\infty}^{+\infty} f dv \; , \tag{11}$$

or by combining Egs.(10) and (11)

$$-(\omega - kv) f = \frac{4\pi q^2}{mk} \frac{\partial f_0}{\partial v} \int_{-\infty}^{+\infty} f dv \; . \tag{12}$$

By deviding by $\omega - kv$ and introducing the plasma frequency $\omega_p^2 = \frac{4\pi e^2 n}{m}$ we obtain

$$\epsilon(k, \omega) \equiv 1 + \frac{\omega_p^2}{nk} \int_{-\infty}^{+\infty} \frac{\frac{\partial f_0}{\partial v} dv}{\omega - kv} = 0 \; . \tag{13}$$

This is the Vlasov dispersion equation for plasma oscillations. It is easily seen that the integral in this relation does not exist in the usual meaning, because of the singularity. Vlasov's prescription was to take the Cauchy principle value. However this was not well founded physically (see also section 5). Several attempts were made to resolve this difficulty, but no one succeeded until Landau resolved the matter in 1946[1].

3 Landau theory

Landaus starting point was to reformulate the problem and solve it as an initial value problem. We shall summarize Landau's solution.

$$\frac{\partial f}{\partial t} + ikvf - i\frac{qk}{m}\phi\frac{\partial f_0}{\partial v} = 0 \; . \tag{14}$$

We take the Laplace transform of this equation by multiplying by $\exp -pt$ and integrate with respect to t from $-\infty$ to $+\infty$ obtaining

$$pf_p + ikvf_p + i\frac{qk}{m}\phi_p\frac{\partial f_0}{\partial v} = g(v)e^{ikx} \quad \text{with} \quad \phi_p = -\frac{4\pi q}{k^2}\int_{-\infty}^{+\infty} dv f_p \; . \tag{15}$$

Here the index p, refers to the Laplace transform and the expression for ϕ_p originates from Eq.(8) and $g(v)e^{ikx} = f(\mathbf{r}, \mathbf{v}, t=0)$. Furthermore we have $kx = \mathbf{k}\cdot\mathbf{r}$, which means that our choice of coordinate system is such that $\mathbf{k}$ is parallel to the x-axis.

From Eqs.(15) we find

$$f_p = -i\frac{qk}{m}\phi_p\frac{\frac{\partial f_0}{\partial v}}{p+ikv} + \frac{g(v)e^{ikx}}{p+ikv} \; . \tag{16}$$

By substituting for f_p from Eq.(16) in Egs.(15) we get

$$\phi_p = i\frac{4\pi q^2}{m}\phi_p\int_{-\infty}^{+\infty}\frac{\frac{\partial f_0}{\partial v}}{p+ikv}dv - \frac{4\pi q e^{ikx}}{k^2}\int_{-\infty}^{+\infty}\frac{g(v)}{p+ikv}dv \tag{17}$$

or

$$\phi_p = -\frac{4\pi i q e^{ikx}}{k^2}\frac{G(k.ip)}{\epsilon(k,ip)} \text{ where } \quad G(ip,k) = \int_{-\infty}^{+\infty}\frac{g(v)dv}{ip-kv} \; , \tag{18}$$

and $\epsilon(k, ip)$ is given in Eq.(13). We then find

$$\phi(x,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\phi_p e^{pt}dp = -\frac{2qe^{ikx}}{k^2}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{pt}dp}{\epsilon(k,ip)}G(k.ip) \; . \tag{19}$$

By introducing new variable $\omega = ip$, Eq.(19) can be rewritten as

$$\phi(x,t) = \frac{2iqe^{ikx}}{k^2}\int_{-\infty+i\sigma}^{\infty+i\sigma}\frac{e^{-i\omega t}}{\epsilon(k,\omega)}G(k,\omega)d\omega \; . \tag{20}$$

Here σ must be chosen acording to usual convention, to the right of all singularities. For the electric field we find

$$E(x,t) = \frac{2qe^{ikx}}{k} \int_{-\infty+i\sigma}^{\infty+i\sigma} \frac{e^{-i\omega t}}{\epsilon(k,\omega)} G(k,\omega) d\omega \ . \tag{21}$$

Since $G(k,\omega)$ is given by initial conditions and $\epsilon(k,\omega)$ is given by the velocity distribution function for equilibrium, we are left with a mathematical problem in function theory, namely to evaluate the integral in Eqs(20) and (21). We notice that apparently we do not get a dispersion equation like what we got in th Vlasov problem, see Eq.(13).

However, if we take the limit $t \to \infty$, then we can also consider this to be a dispersion equation. In the case where $G(k,\omega)$ is an analytic function with regard to ω for $\sigma > 0$ then the asymtotic form for $E(x,t)$ for large t will be determined by the zeros in the denominator. Let these be given by $\omega = \omega_k$, or

$$\epsilon(k,\omega_k) = 0 \ , \tag{22}$$

which results in solutions

$$E(x,t) \sim \exp i(kx - \omega_k t) \ , \tag{23}$$

and we obtain plasma oscillations compared to Eq.(13). But since the integration over ω takes place along a path in the upper half plane this has to be taken into consideration when $\epsilon(k,\omega)$ is computed using Eq.(13), this means that

$$\frac{1}{\omega - kv} = \frac{1}{\omega_r + i\omega_i - kv} \ ,$$

and when pasing to the limit $\omega_i \to 0$ we have Plemelj's formulae

$$\lim_{\omega_i \to +0} \frac{1}{\omega - kv} = \frac{P}{\omega - kv} - i\pi\delta(\omega - kv) \tag{24}$$

where now P means taking the principal value and $\delta(x)$ is Dirac's delta function. This way to pass around the pole is named Landau's rule. Making use of this Eq.(13) can be worked out to give

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{nk} P \int_{-\infty}^{\infty} \frac{\frac{\partial f_0}{\partial v} dv}{\omega - kv} - i\pi \frac{\omega_p^2}{n|k|k} \frac{\partial f_0}{\partial v} \bigg|_{v=\frac{\omega}{k}} \tag{25}$$

where $|k|$ comes from integration over Dirac's delta function. For ω complex this can also be written as

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{nk} \int_{-\infty}^{\infty} \frac{\frac{\partial f_0}{\partial v} dv}{\omega - kv} \ , \tag{26}$$

which has Eg.(25) as the limiting case $\Im\omega \to 0$.

Even if Eq.(21) formally represent the solution to the problem at hand, it contains, even for isotropic velocity distribution functions, three integrals. (Notice that the definition of f_0, Eq.(9), contains a double integral. These integrals can only be solved analytical for the most trivial cases, like step functions delta functions and "resonance functions" $\sim (v^2 + a^2)^{-n}$. For instance Maxwell distribution functions are not tractable. However, we still can find considerable information from Eq.(21) in the limiting case $t \to \infty$.

We shall now pay some more attention to the case where f_0 is given by a Maxwell distribution

$$f_0 = \frac{n}{\sqrt{\pi} v_e} e^{-\frac{v^2}{v_e^2}} , \tag{27}$$

where v_e is the thermal mean velocity of electrons. Eq.(25) now takes the form

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{kv_e} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{-\frac{2v}{v_e^2} e^{-\frac{v^2}{v_e^2}} dv}{\omega - kv} \quad . \tag{28}$$

By introducing new variables $y = \frac{v}{v_e}$ and $\eta = \frac{\omega}{kv_e}$ we find

$$\epsilon(k,\eta k v_e) = 1 + \frac{\omega_p^2}{k^2 v_e^2} \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{y e^{y^2}}{y - \eta} dy = 1 + 2\frac{\omega_p^2}{k^2 v_e^2} \Big\{ 1 + \frac{\eta}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{y^2}}{y-\eta} dy \Big\} =$$

$$1 + 2\frac{\omega_p^2}{k^2 v_e^2} \{1 + \eta Z(\eta)\} \tag{29}$$

where we have introduced the plasma dispersion relation [8]

$$Z(\eta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{y^{-2}}}{y - \eta} dy. \tag{30}$$

This function is discussed and tabulated by Fried and Conte 1961, it can also be written as

$$Z(\eta) = i\sqrt{\pi} e^{-\eta^2} \{1 + \mathrm{erf}(i\eta)\}. \tag{31}$$

Notice that for complex values of η the integral in Eq.(29) always exists, and for real values we have to take the limit $\Im\eta \to 0$, which means that in the formula, Eq.(25), we have to substituted for f_0, by f_0 in Eq.(27). We shall now study Eq.(29) with $\epsilon(k, \eta k v_e) = 0$ in the limit $\eta \to \infty,\ k \to 0$.

For $\Im\eta \simeq 0$ we find (compare Eq.(25))

$$Z(\eta) = \frac{1}{\sqrt{\pi}} P \int_{-\infty}^{\infty} \frac{e^{-y^2}}{y - \eta} dy + i\sqrt{\pi} e^{-\eta^2} \tag{32}$$

where the last contribution comes from the pole in the integrand. By series expansion of the denominator in the integral with respect to y/η, we find

$$Z(\eta) = -(\frac{1}{\eta} + \frac{1}{\eta^2} + n\frac{1}{2\eta^3} + \frac{3}{4\eta^5} + \frac{15}{8\eta^7} + \cdots) + i\sqrt{\pi} e^{-\eta^2}. \tag{33}$$

This is an asymptotic series with respect to large $|\eta|$. We now write $\frac{\omega_p}{kv_e} = \eta_0$ and obtain the following result from Eq.(29)

$$1 + 2\eta_0^2 \{1 - (1 + \frac{1}{2\eta^2} + \frac{3}{4\eta^4} + \frac{15}{8\eta^7} + \cdots)\} + 2i\sqrt{\pi} \eta \eta_0^2 e^{-\eta^2} = 0. \tag{34}$$

One can then show that the first significant terms in this asymptotic series for large $|\eta|$ are

$$1-\frac{\eta_0^2}{\eta^2}+2i\sqrt{\pi}\eta_0^2\eta e^{-\eta^2}=0. \qquad (35)$$

To lowest order we have that η is real and equal η_0. We then write $\eta=\eta_0+i\eta_i$, solve with respect to η_i, and find

$$\eta_i=-\sqrt{\pi}\eta_0^4 e^{-\eta_0^2}. \qquad (36)$$

We continue writing $\omega=\eta k v_e$ with

$$\omega=\omega_0-i\gamma \qquad (37)$$

to obtain

$$\omega_0\approx\pm\omega_p \qquad \gamma\approx\sqrt{\pi}\frac{\omega_p^4}{k^3v_e^3}e^{-\frac{\omega_p^2}{k^2v_e^2}}. \qquad (38)$$

Introducing the Debye-length λ_D by

$$\lambda_D=\sqrt{\frac{T}{4\pi\eta_0e^2}}=\frac{1}{\sqrt{2}}\frac{v_e}{\omega_p} \qquad (39)$$

we find

$$\gamma\approx\sqrt{\frac{\pi}{8}}\frac{\omega_p}{\lambda_D^3k^3}e^{-\frac{1}{\lambda_D^2k^2}}. \qquad (40)$$

This is Landau's result. The condition for this approximation is $k\lambda_D<<1$.

We notice that in this approximation the damping is exponential small. From the complete solution for the electric field, Eq.(21), we see that for reasonable large values of t we get a contribution from the pole, $\omega-kv$, in the expression for $G(\omega,k)$, Eq.(18). This correspond to a spread of particles with respect to the perturbation from which we started. This of course implies that the evolution of the electric field is very complicated as time passes on.

4 Physical interpretation of Landau damping

Many have tried to explain Landau damping from simple physical models. see T. Dawsen 1961 [2]Dawsen, Ching-Sheng Wu (1962)[5] and Montgomery & Tideman (1964)[6].

The main focus on all of these models is to concentrate on particles being in resonance with the "electrostatic" wave propagating through the plasma. Looking at Eq.(25) it is clear that in the general case the origin of the imaginer part of ω is proportional to $\frac{\partial f_0}{\partial v}$ for particles moving close to phase velocity ω/k.

This supports the following interpretation: Those particles moving a little bit slower than the wave will be accelerated and those moving a little faster will be retarded. The first group of particles thus will takes away energy from the wave and the second group gives away energy to the wave. Therefor if $\frac{\partial f_0}{\partial v}>0$ there will be a net transfer of energy from the wave to the particles, in other words a *damping.*

We can also view this problem from a different angel. It can be shown that the distribution function continues to oscillate for all time, however integrated quantities like the electric field become damped because of phase mixing from different parts of the distribution function. This topic is discussed in section 4.

5 Van Kampen waves

Following the historical path, the next essential contribution was given by Van Kampen 1955[10]. Van Kampen showed that the usual technique of *normal modes* used by Vlasov [**?**] could be extended to give a complete solution in agreement with Landau's result.

We give a brief overview by returning to Eq.(10). When we solved this equation we divided by the factor $(\omega - kv)$, which could be zero. It is therefor clear that to the solution we have found, we can add the solution to the equation

$$(\omega - kv)f = 0 \;, \tag{41}$$

which have the nontrivial solution

$$f = \lambda(v)\delta(\omega - kv) \;. \tag{42}$$

Therefore the full solution of Eq.(10) not only contains the discrete spectrum of normal modes, but in addition contains a continues spectrum facilitated by the Dirac delta function solution given in Eq.(42), such that the complete solution can be written as

$$f = -\frac{4\pi q^2}{mk}\frac{\frac{\partial f_0}{\partial v}}{\omega - kv}P\int_{-\infty}^{\infty} f dv + \lambda(v)\delta(\omega - kv) \;, \tag{43}$$

where P again means we integrate over the singularity $(\omega - kv)$ by taking the Cauchy principal value. Notice we now have included the contribution from the singularity in the delta function term. Otherwise Eqs. (42) and (43) tells us that $\lambda(v)$ is completely arbitrary. But since Eq.(10) is homogeneous in f, we can normalize such that

$$\int_{-\infty}^{\infty} f dv = 1 \;.$$

This way we obtain

$$\lambda(\frac{\omega}{k}) = |k|\{1 + \frac{4\pi q^2}{mk}P\int_{-\infty}^{\infty}\frac{\frac{\partial f_0}{\partial v}}{\omega - kv}dv\} \;, \tag{44}$$

comparing now to Eq.(25) we observe that if we take

$$\lambda(\frac{\omega}{k}) = -i\pi\frac{\omega_p^2}{n|k|k}\frac{\partial f_0}{\partial v}\bigg|_{v=\frac{\omega}{k}} \;, \tag{45}$$

this equation becomes identical to Eq.(25). We also notice that the extra contribution to f in Eq.(43) comes from a tiny region around $v = \omega/k$.

It is now possible to show that the solutions found this way are complete in the sense that any perturbation can be expanded in this class of solutions, this was done by Van Kampen [10] and later Case 1959[3] could show that Landau's and Van Kampen's results were identical.

Still Backus[2] has commented on this, and pointed out that Van Kampen's result relay on the fact that the distribution function $f_0(v)$ is stable, which is not explicitly referred to by Van Kampen. So it is only for this case the Van Kampen solutions represent a complete set. Backus has also shown that the Laplace transform technique as used by Landau, always gives the right answer, because no instability grows faster than exponentially. And therefore the inverse Laplace transform always exists.

We shall now take a look at the physical implications of Van Kampen's results. As already mentioned, λ in Eq.(42) is arbitrary, which means that Eq.(43) is not a dispersion equation like $\epsilon(\omega, k) = 0$, but an equation that for given ω and k determines λ, thus it follows that for a given k, ω can be arbitrary. From Eq.(43) we see that λ can be interpreted as a ray of particles moving with the phase velocity of the wave. So with a carefully chosen perturbation we can have plasma waves propagating undamped. Because we can choose λ such that the last term in Eq.(25), the term giving Landau damping, cancel. Physically this means that we from the starting point add to the perturbation a stream of particles moving with the phase velocity of the wave and designed so that there is no net energy transfer from particles to the wave.

This is the so called Van Kampen waves, which can be interpreted as a stream of particles moving with the phase velocity ω/k.

The reason these waves were absent in Eq.(25) was that we required certain regularity conditions on the distribution function at $t = 0$, namely that it should be a whole function, see Jackson 1960[11].

6 Critical remarks

By such a short, summarizing and superficial account as we have given here one can easily end up believing that these problems basically are understood and solved. But that is not true, there still are many aspects of this topic that are not understood or resolved. The first question is: How well will a linear theory describe such phenomenon? The second question: When will nonlinear effects become important? See Eq.(4). We shall now pay some attention to these issues in the following account.

A third question: Hayes[12] has pointed out that in addition to the zero-points for $\epsilon(k, \eta k v_e) = 0$ as localized in Eqs.(29) and (30), it must exist infinitely many zero-points due to Picard's theorem. It is still an unresolved problem localizing these zero-points. The positioning of these zero-points could strongly affect the time it takes before one enters the asymptotically damped regime.

Finally there exist tricky problem of a priori assuming the distribution function to be an analytic function at time $t = 0$. Also it is not easily understood how such unphysical technicalities as the behavior of the distribution function for complex arguments should influence the behavior of the electric field, for instance.

On the other side there is a close correspondence in between the behavior of a function for real arguments and complex arguments. These problems have been thoroughly examined in several works by H. Weitzner, 1963[13], 1964[14] and 1965[15]. The same author[16] also study ion waves in a collision-less plasma, where both electron and ion-oscillations are considered. We will now pay attention to some experimental results regarding this topic.

7 Experiments with Landau damping

It took nearly 20 years, from Landau's work[1] in 1946 before the first experiments with believable results were done to test this theory. These were done by Malmberg et. al. in 1964[17] and 1966[25]. Even though some work was done, Wong at.al., 1964[19], where observations of wave-phenomenon in the ionosphere was indicating the existence of Landau damping of ion acoustic waves, we agree with Kadomtsev 1968[20], when he says that we have to look at the experiment by Malmberg et.al. as the first direct experimental evidence for the existence of Landau damping as phenomena.

In the firs experiment it is shown without doubt how damping is dependent on particles in the distribution function that move with the phase velocity of the wave. This setup was in the framework of a boundary value problem and the observed phenomenon was Landau damping i space.

Later experiments by Malmberg and Wharton, 1966 and 1967, have confirmed these results in detail, and we can conclude that the dispersion equation

$$\epsilon(k, \omega) = 0$$

has been verified.

8 Nonlinear Landau damping

It is very interesting to investigate this problem for wave amplitudes large enough that nonlinear effects become important. To some extent one may say that the linear problem basically is solved by Landau at least in the limit $t \to \infty$. Corresponding statements can not be made about the nonlinear case, that is the problem we have when keeping the term given by Eq.(4), a term omitted in Vlasov's eguation, Eq.(5).

Still a great deal of work has been done studying finite amplitude waves looking for collision-less damping. One technique used is the so called quasi-linear approach where one make an expansion with respect to Wave modus and look at coupling in between different modes and treat the mode-coupling terms as small, using straight forward perturbation techniques. The pioneering work on this topic was done by Drummond and Pines, 1962[21]. O'Neil 1965[22] also uses this technique in the study of collision-less nonlinear damping of plasma oscillations.

It will be to farfetched to go into details of the Vlasov nonlinear problem here. Instead we will make brief comments on the problem as seen from a phonological point of view. This is because it provides some insight into the physics behind such processes. We will mostly follow an overview article by Kadomtsev 1968[20].

We study an "electrostatic" wave having the potential $\phi = \phi_0 \cos(\omega_0 t - kx)$. Furthermore the amplitude ϕ_0 is considered to be small and the phase velocity $v_f = \omega_o/k >> \sqrt{2T/m}$ (that is the thermal speed), this means there is a relatively small number of electrons having a speed around the phase velocity v_f, in other words we are on the tail of the distribution function. We can therefore look at ϕ being nearly constant for the time interval considered, because so few particles are interacting with the wave. Thus we are studying the behavior of particles in resonance with the wave having nearly constant amplitude. If we move to a frame of reference following the wave, the electrons will experience a potential $\phi = \phi_0 \cos(kx)$. The electrons can be conveniently divided in two groups. This is the electrons being trapped in the potential which oscillates in between two maxima, and the second group is the electrons having sufficient energy, (are sufficiently different in speed compared to $v = v_f$) to pass the potential hilltops, i.e. the transient electrons.

From the Newton's law we have

$$-eE = e\frac{\partial \phi}{\partial x} = m\ddot{x} \ , \quad -e \text{ is electron charge} \tag{46}$$

or

$$\frac{d^2 x}{dt^2} + \frac{\phi_0 k e}{m} \sin(kx) \ . \tag{47}$$

For small amplitude oscillations we have approximately harmonic oscillations with frequency

$$\Omega = k\sqrt{\frac{e\phi_0}{m}} \tag{48}$$

Studying Eq.(47) in phase-plane $x, y = \frac{dx}{dt}$ we find

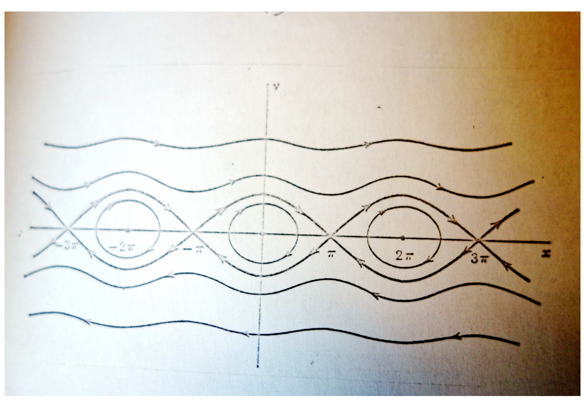

Figure 1:

Here we have that ϕ has a max at $x = 0, \ \pm 2\pi, \ \cdots$. Notice that the potential energy for electrons is $-e\phi$, which has its minimum at the same place as ϕ. The separatrix in Fig.1 separate the trapped and transient particles.

The frequency decreases when we move from the center $(x = 0,\ v = 0)$ outwards. Particles being located in $v = 0,\ x = \pm(2n + 1)\pi, \quad n = 1, 2, 3 \cdots$ have the speed zero and no acceleration because the force is zero at this point. Therefore these particles are at rest. From a continuity point of view we see that particles close to the separatrix have a small mean velocity whether they are trapped or transient. The total energy for a particle

$$\frac{m(v - v_f)^2}{2} - e\phi = \text{constant}. \tag{49}$$

Since $(v - v_f)|_{x=\pm\pi} = 0$ and $\phi(\pm\pi) = 0$, we have

$$\frac{1}{2}m(v - v_f)^2 = e\phi \quad \text{and} \quad \frac{2e}{m}\phi_0(v - v_f)^2_{max} \ . \tag{50}$$

The half-width of the separatrix at $x = 0$ now is

$$\Delta v = \sqrt{\frac{2e\phi_0}{m}} \ . \tag{51}$$

This shows that the Δv decreases with amplitude, but slower than linear. Further it shows that the number of trapped particles can be relatively large even for a small wave amplitude.

Now we try to follow the time-evolution of the distribution function F. Since the interaction-region $2\Delta v$ is relatively broad we will omit the perturbation in the distribution function, since this is a higher order effect. Formally we can write

$$\mathbf{r} = \{x, v\} \quad \text{and} \quad \mathbf{u} = \frac{d\mathbf{r}}{dt} = \{\frac{dx}{dt}, \frac{dy}{dt}\} \quad \text{or} \quad \mathbf{u} = \{v, \frac{e}{m}\frac{\partial\phi}{\partial x}\} \ , \tag{52}$$

and we can write the Vlasov equation as

$$\frac{\partial F}{\partial t} + \mathbf{u}\cdot\nabla F = 0 \ , \quad \nabla \equiv \frac{\partial}{\partial \mathbf{r}} \ , \quad \nabla\cdot\mathbf{u} = 0 \ . \tag{53}$$

Thus we can regard Eq.(53) as the equation of continuity for a substance F streaming in the phase-plan (x, v), and since $\nabla \cdot \mathbf{u} = 0$, this flow is incompressible and F is conserved along a stream line. From this we can make up a picture of the variation in F, when we know the flow-pattern and F, at the starting point.

In Fig. 2 we shadowed the region corresponding to trapped particles, those being inside the separatrix, and therefore are in resonance with the wave. Being in resonance these particles oscillate with a frequency close to Ω in the potential of the wave and will therfore start to rotate in the phase-plane. This is shown in the left part of Fig.2 (after Kadomtsev[20]). In Fig.2 a) we have the start situation, in Fig.2 b) the picture after one half period and in c) after several periods. We therefore have that the particles to the left of v_f in Fig.2 change place after one period. If there are equally many particles to the left and right of v_f there will be no change, but if $\frac{\partial F}{\partial v}|_{v=v_f} \neq 0$, this is not the case. After half a period we have the situation shown in Fig.2 b), and if the problem was linear we would be exactly back to the starting position. But Ω is not the same for different particle groups as we move away from the cener of the phase-plane. This will result in a mixing of particles, a phase mixing, such

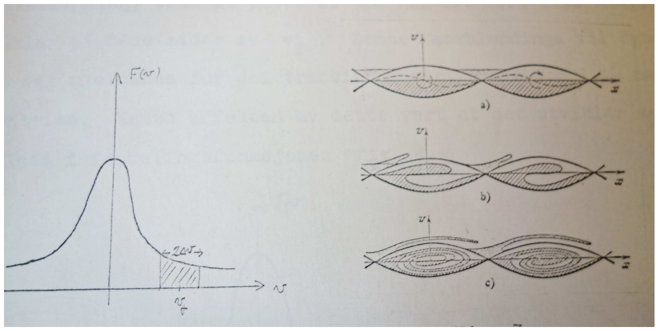

Figure 2:

that after some time there will be equally many particles on each side of v_f. The net effect being that this process will produce a plateau in the distribution function. See Fig.3

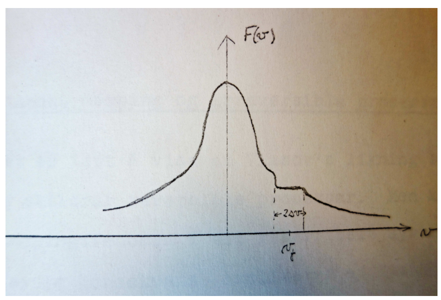

Figure 3:

The wave we obtain after this process is finished corresponds to the stationary solution of Vlasov's equation. This is because at this point the singular point $(\omega - kv) = 0$ in Eq.(13) do not make any contribution since $\frac{\partial f_0}{\partial v}|_{\omega=kv} = 0$, and it becomes meaningful integrating Eq.(13) and taking the principal value of the integral. The stationary case can therefore be realized physically as the evolution of a wave with finite amplitude.

We can also look at this problem from the angel of Van Kampen modes. If we from the starting point add a "ray" of particles such that the distribution function from the start have a plataue at $v = \frac{\omega}{k}$, then undamped oscillations can exist. In this situation phase mixing will not have any other effect than preserving the already existing plataue.

Another intersting aspect with these equations are that they are reversible. The question then is: How can a reversible process be consistent with damping? And can these processes be examined experimentally? We will touch these issues in the next section.

9 Landau damping and irreversible processes, echo phenomenon

It is easy to show that Vlasov's equation together with Poisson's equation govern reversible processes. But how do you interpret damping in a reversible system? Reversible in the strict sense means that the system conserves the memory of its initial state for all time. And if we at any time enters the system and reverse the velocity for all the particles, the system would evolve towards the initial state at $t = 0$. Such a process can of course never be realized in practical terms. It is therefore very difficult to test whether an actual system is reversible. Still this can be done, and it is within existing experimental techniques possible to observe effects that can measure reversibility of a system. We are now talking about the so called *echo phenomenon*. We will give a short summary of results in this respect, essentially following work by Gould et.al.[23] 1967.

Echo phenomenon have been known for a long time in connection with resonance effects in nuclear physics Hahn 1950[24]. But echo phenomenon in plasma physics is a more recent field of research. From the Landau's theory we have learned that macroscopic quantities like the electric field and charge density can be damped exponentially whereas the distribution function in phase space continues to oscillate for ever. In this connection one may think of this effect as an integrated effect in phase space because as time passes on these oscillations becomes faster and faster. The distribution function takes the form

$$f_2(v, x, t) = f_1(v) e^{-ik_1 x + ik_1 t} \; . \tag{54}$$

Therefore as time passes on the interval in velocity space $\Delta v = \frac{2\pi}{k_1 t}$, corresponding to one period in the oscillations becomes smaller and smaller. This is the cause for integrated quantities like the electric field and charge density to become damped. This is also the reason for this effect to be called *phase mixing*.

Echo phenomenon in plasma are related to well known echo phenomenon and have its origin in reversing a damping process which is due to phase mixing, by interacting with the system and reversing the phase-evolution on the microscopic level.

First we give a simple physical explanation. Let us think that we at time $t = 0$ interact with the system to perturb it with an electric field $\sim e^{-ik_1 x}$, then this field becomes damped and leave behind a distribution function like the one given by Eq.(54). We then interact with the system at time $t = \tau$ with the perturbation $\sim e^{ik_2 x}$. This will leave a first order perturbation in the system of the following form

$$f_2(v) = e^{ik_2 x - ik_2 v(t-\tau)} \; . \tag{55}$$

In addition to this the first order perturbation that already existed will be modulated with a second order term

$$f_(v) f_2(v) = e^{i(k_1-k_2)x + i\{k_2\tau + (k_1-k_2)t\}v} \; . \tag{56}$$

We see that at a certain time $t = \frac{k_2}{k_1-k_2}\tau$ the coefficient of v in the exponent becomes zero. The integral over the distribution function will not be zero due to phase mixing at this point. If τ is large compared to the characteristic time for collision-less damping and $\frac{k_2}{(k_1-k_2)} \sim 1$, then this third electric field will show up long after the first wave is being damped to a very low level by phase mixing. This means we have an *echo*. Such echo phenomenon naturally are interesting by itself because they could give some measure of the collision frequency in the plasma. This is because collisions will wipe out the information otherwise being stored in the collision-less plasma and thus reveal to what extent collisions play a role for the actual plasma-medium. Therefore it is of great interest to study systems where collisions were not neglected and find what influence they have on the echo effect.

The echo phenomenon can be studied rigorously based on Vlasov's and Poisson's equations. In the following we study a one dimensional system, and assume perturbations at $t=0$ and $t=\tau$ by two pulses in the electric potential

$$\phi_{ext} = \Phi_{k_1}\cos(k_1 x)\delta(\omega_p t) + \Phi_{k_2}\cos(k_2 x)\ \delta\{\omega_p(t-\tau)\}\ , \qquad (57)$$

where we have put in the plasma frequency, ω_p, in the argument for the delta-function in order to make it dimension-less.

We now apply Fourier transform in space and Laplace transform in time of the Vlasov - Poisson system of equations. We find equations to first and second order when expanding with respect to amplitude in the imposed pulses. The theory otherwise is the usual quasi linear theory, see Drummond and Pines 1962[7].

$$\tilde{f}_k^{(1)}(p) = \frac{e}{m} ik\tilde{\phi}_k^{(1)}(p)\frac{\frac{\partial f_0}{\partial v}}{p+ikv} \qquad (58)$$

$$\tilde{f}_k^{(2)}(p) = \frac{e}{m} ik\tilde{\phi}_k^{(2)}(p)\frac{\frac{\partial f_0}{\partial v}}{p+ikv} + \frac{e}{m}{\sum_q}' \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\phi p'}{2\pi i} i(k-q)\tilde{\phi}_{k-q}^{(1)}(p-p')\frac{\frac{\partial \tilde{f}_q^{(1)}(p')}{\partial v}}{p+ik} dp'\ , \qquad (59)$$

(Notice that the prime on the summation sign means that we do not count $q=0$).

$$k^2\tilde{\phi}_k^{(1)}(p) = 4\pi ne\int_{-\infty}^{\infty} dv\tilde{f}_k^{(1)}(v,p) + \frac{k_1^2\Phi_{k_1}}{2\omega_p}\{\delta_{k,k_1}+\delta_{k,-k_1}\} + \frac{k_2^2\Phi_{k_2}}{2\omega_p}\{\delta_{k,k_2}+\delta_{k,-k_2}\}e^{-p\tau} \qquad (60)$$

$$k^2\tilde{\phi}_k^{(2)}(p) = 4\pi ne\int_{-\infty}^{\infty} dv\tilde{f}_k^{(1)}(v,p)\ . \qquad (61)$$

For a given quantity q, $\tilde{q}$ means the Laplace transformed of q and the index k refers to the Fourier transform. We then obtain

$$k^2\tilde{\phi}_k^{(2)} = \omega_p^2\tilde{\phi}_k^{(2)}(p)\int_{-\infty}^{\infty} ik\frac{\frac{\partial f_0}{\partial v}}{p+ikv}dv + \int_{-\infty}^{\infty} dv\left\{\frac{e}{m}\sum_q\int_{\sigma'-i\infty}^{\sigma'+\infty}\frac{2dp'}{4\pi i} i(k-q)\tilde{\phi}_{k-q}^{(1)}(p-p')\frac{\frac{\partial \tilde{f}_q^{(1)}(p')}{\partial v}}{p+ikv}\right\}. \qquad (62)$$

From Eqs.(58) and (60) we find

$$\tilde{\phi}_k^{(1)}(p) = \frac{4\pi ne}{k^2}\int \frac{e}{m}\tilde{\phi}_k^{(1)}(p)\frac{\frac{\partial f_0}{\partial v}}{v+\frac{p}{i(k-q)}}dv + \frac{k_1^2\Phi_{k_1}}{2k^2\omega_p}\{\delta_{k,k_1}+\delta_{k,-k_1}\} + \frac{k_2^2\Phi_{k_2}}{2k^2\omega_p}\{\delta_{k,k_2}+\delta_{k,-k_2}\} \tag{63}$$

from which we obtain

$$\tilde{\phi}_k^{(1)}(p) = \frac{1}{\epsilon(k,ip)}\left[\frac{k_1^2}{k^2}\frac{\Phi_{k_1}}{2\omega_p}\Big\{\delta_{k,k_1}+\delta_{k,-k_1}\Big\} + \frac{k_2^2\Phi_{k_2}}{k^2 2\omega_p}\{\delta_{k,k_2}+\delta_{k,-k_2}\}e^{-p\tau}\right] \tag{64}$$

where $\epsilon(k, ip)$ is given by Eq.(13). Integrating by parts we find

$$\tilde{\phi}_k^2(p) = \frac{\omega_p^2}{ik\epsilon(k,ip)}\frac{e}{m}\int_{-\infty}^{\infty} dv \sum_q{}' \int_{\sigma'-\infty}^{\sigma'+\infty}\frac{dp'}{2\pi i} i(k-q)\tilde{\phi}_{k-q}^{(1)}(p-p')\frac{\tilde{f}_q^{(1)}(p')}{(p+ik)^2}\ . \tag{65}$$

By substituting from Eq.(58) in Eq(65) we obtain

$$\tilde{\phi}_k^2(p) = \frac{\omega_p^2}{ik\epsilon(k,ip)}\frac{e}{m}\int_{-\infty}^{\infty} dv \sum_q{}' \int_{\sigma'-\infty}^{\sigma'+\infty}\frac{dp'}{2\pi i} i(k-q)\tilde{\phi}_{k-q}^{(1)}(p-p')\frac{e}{m}iq\tilde{\phi}_q^{(1)}(p')\frac{\frac{\partial f_0}{\partial v}}{(p'+iqv)(p+ik)^2}\ . \tag{66}$$

Fore $k_3 = k_2 - k_1$, $\quad q_1 = -k_1$ and $q_2 = k_2$ we find that

$$\sum_q{}' \frac{(k-q)q}{p'+iqv}\ \tilde{\phi}_{k-q}^{(1)}(p-p')\ \tilde{\phi}_q'(p') = -k_1k_2\left[\frac{\tilde{\phi}_{k_2}^{(1)}(p-p')\ \tilde{\phi}_{-k_1}^{(1)}(p')}{p'-ik_1v} + \frac{\tilde{\phi}_{-k_1}^{(1)}(p-p')\ \tilde{\phi}_{k_2}^{(1)}(p')}{p'+ik_2v}\right]. \tag{67}$$

From Eqs.(66), (67) we now find

$$\tilde{\phi}_{k_3}^{(2)} = \frac{1}{2\pi i}\int_{\sigma-\infty}^{\sigma+\infty} e^{pt}\ \tilde{\phi}_{k_3}(p)dp \quad =$$

$$\frac{1}{2\pi i}\int_{\sigma-\infty}^{\sigma+\infty} dp \frac{\omega_p^2 e/m}{ik_3\ \epsilon(k_3,ip)}\int_{-\infty}^{\infty} dv \int_{\sigma'-\infty}^{\sigma'+\infty}\frac{dp'}{2\pi i}(-k_1k_2)\frac{\frac{\partial f_0}{\partial v}}{(p+ik_3v)^2}\frac{\Phi_{k_1}\Phi_{k_2}}{4\omega_p^2}\cdot A\ , \tag{68}$$

where

$$A \equiv \left[\frac{e^{pt}\ e^{-p'\tau}}{\epsilon(k_2,ip')\ \epsilon\Big(-k_1, i(p-p')\Big)(p'+ik_2v)} + \frac{e^{p(t-\tau)}\ e^{p'\tau}}{\epsilon(-k_1,ip')\ \epsilon\Big(k_2, i(p-p')\Big)(p'-ik_1v)}\right]$$

or

$$\phi_{k_3}^{(2)} = \frac{e}{m}\frac{\Phi_{k_1}\Phi_{k_2}ik_1k_2k_3}{4k_3^2}\int_{-\infty}^{\infty} dv \int_{\sigma-i\infty}^{\sigma+i\infty}\frac{dp}{2\pi i}\int_{\sigma'-i\infty}^{\sigma'+i\infty}\frac{dp'}{2\pi i}\ \frac{A\ \frac{\partial f_0}{\partial v}}{\epsilon(k_3,ip)(p+ik_3)^2}\ . \tag{69}$$

When integrating over p and p' we can always choose $0 < \sigma' < \sigma$. In order to evaluate the integrals we use the usual residue-calculus. We will omit the contributions from the three dielectric functions $\epsilon(k, ip)$. This is motivated by the fact that we are interested in the solutions where values of τ fulfill the requirements $|\gamma(k_1\tau|,\ \ |\gamma(k_2\tau|,\ \ |\gamma(k_3\tau| >>\ 1$, where $\gamma(k)$ is the Landau damping constant at time $t = 2\tau$.

This means that the time passed from the last pulse to an eventually echo is of the order τ, which means that the electric fields connected to each of these pulses are damped and disappeared.

We only get contributions from the residues associated with the three poles $p = ik_1v$, $p = -ik_2v$ and $p = ik_3v$. First we integrate over p' by closing the contour such that the integrand approach zero on a half circle centered in $(\sigma', 0)$.

Looking at the first term in A, we do the integration by closing the contour in the right half-plane and for the last term in the left half-plane. Since all the singularities are located on the imaginary axis, having $\sigma > 0$ there will be no singularities from poles inside the contour in the right half-plane. From integration in the left half-plan we obtain

$$\int_{\sigma' - \infty}^{\sigma , +\infty} \frac{dp'}{2\pi i}\ A\ = \frac{e^{p(t-\tau)}\ e^{ik_1 v\tau}}{\epsilon(-k_1, -k_1v)\ \epsilon(k_2, ip + k_1v)}\ . \tag{70}$$

One can easily achieve integration over p by doing a series expansion of $e^{p(t-\tau)}$ around $p = -ik_3$, in order to obtain the residue at this point (we have to use the that $t > \tau$ in order to get any contribution), and we obtain the final result

$$\tilde{\phi}^{(2)}_{k_3} \approx \frac{e}{m}\frac{\Phi_{k_1}\ \Phi_{k_2}\ k_1k_2\ ik_3(t-\tau)}{4k_3^2}\int_{-\infty}^{\infty} dv \frac{\frac{\partial f_0}{\partial v} e^{-ik_3(t-\tau)v + ik_1v\tau}}{\epsilon(k_3, k_3v)\ \epsilon(-k_1, -k_1v)\ \epsilon(k_2, k_2v)} \tag{71}$$

We now observe that for $k_3(t-\tau) \approx k_1\tau$ or at $t \approx \tau, \equiv \frac{k_1+k_3}{k_3}\tau = \frac{k_2}{k_3}\tau$, there is no phase-mixing and $\tilde{\phi}^{(2)}_{k_3}$ gets a finite value or an *echo* at this point in time.

We have given a brief account of *echo* phenomenon in time, but one can also study this phenomenon in space. If exploring a boundary-value problem with damping in space, it can be shown to exist corresponding *echo* in space. This means that at a specific position in space we obtain an *echo* in space.

This has been tested experimentally by Malmberg[25] et.al. 1968. At this time one think of *echo*-phenomenon in plasma to be experimentally verified. The results regarding these kind of phenomenon have to a great extent broaden our understanding of $Landau - damping$.

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